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# CONTROLLED INVARIANT DISTRIBUTIONS FOR AFFINE SYSTEMS ON MANIFOLDS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 93C10, 53C40

Controlled invariant distributions for affine systems on manifolds *)

by

H. Nijmeijer

ABSTRACT

The purpose of this paper is to give an exposition of a new approach to the problem of nonlinear (A,B)-invariance. We will introduce this problem through the concept of distributions. With the ideas of the geometric approach to linear systems in mind we will derive the solution of this problem under conditions which are equivalent to those in the linear situation.

KEY WORDS & PHRASES: *Nonlinear system theory, (involutive) distributions, linearizable systems, controlled-invariant distributions*

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*) This paper has been presented at the Mathematical Systems Theory Meeting, University of Warwick, U.K., 7th-11th July, 1980.

## 1. INTRODUCTION

The geometric approach for linear systems is a successful way to solve various synthesis problems in control theory, for example the Disturbance Decoupling Problem (D.D.P.) and other, related decoupling problems (see e.g. [16]. It would be interesting to develop an analogue theory for nonlinear systems. Apparently at this moment differential geometry is the adequate apparatus (see e.g. [7],[8]). In this paper we want - by using differential geometry - to discuss nonlinear (A,B)-invariance. Here we don't consider output-maps, for reason that we don't need them in defining (A,B)-invariance, although it will be clear that for synthesis problems one also has to bring in output-maps. The systems that will be treated here have the form (locally) $\dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} u_i(t)B_i(x(t))$.

## 2. PRELIMINARIES ON DIFFERENTIAL GEOMETRY

In this section we give a brief review of the necessary parts of the theory of calculus on manifolds. The reader is referred to BOOTHBY [1], SPIVAK [2] and especially for the analytic case to VARADARAJAN [3]. Sometimes we have to distinguish between the analytic ($C^{\omega}$.) and the smooth ($C^{\infty}$.) case for the differences that will appear.

We start with a $C^{\infty}$ n-dimensional manifold M. By $C^{\infty}(M)$ we denote the collection of all $C^{\infty}$ functions on M and also $V^{\infty}(M)$ will be the collection of all $C^{\infty}$ vectorfields on M. The tangentspace of M will be denoted by TM and in a point m by $T_mM$.

<u>DEFINITION 2.1</u>. The Lie-bracket of two $C^{\infty}$ vectorfields $X,Y \in V^{\infty}(M)$ is another vectorfield, denoted by $[X,Y]$, and defined by

$$[X,Y](f) = X(Y(f)) - Y(X(f)) \qquad \forall f \in C^{\infty}(M)$$

or briefly

$$[X,Y] = XY - YX.$$

In local coordinates (i.e. in a chart of M) the bracket can easily be computed: If

$$X(x) = \sum_{i=1}^{n} X^i(x) \frac{\partial}{\partial x_i}\Big|_x \quad \text{and} \quad Y(x) = \sum_{i=1}^{n} Y^i(x) \frac{\partial}{\partial x_i}\Big|_x$$

then

$$[X,Y](x) = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} X^i(x) \frac{\partial Y^j}{\partial x_i}(x) - Y^i(x) \frac{\partial X^j}{\partial x_i}(x) \right) \frac{\partial}{\partial x_j}\Big|_x.$$

Next we will give a definition which turns out to be one of the most important concepts of this paper:

DEFINITION 2.2. A k-dimensional distribution $\Delta$ on M ($C^\infty$-manifold) is a map $\Delta$ which assigns to each point $m \in M$ a k-dimensional subspace of $T_m M$. The distribution is called $C^\infty$ if for all $m \in M$ there exist a neighbourhood $U(m)$ of m and $X_1,\ldots,X_k \in V^\infty(M)$ such that for each point p in $U(m)$: $\mathrm{Span}\{X_1(p),\ldots,X_k(p)\} = \Delta(p)$. We note that $X_1,\ldots,X_k$ are linear independent in each point of $U(m)$.

REMARK. By $X \in \Delta$ for a vectorfield X we mean that $X(p) \in \Delta(p)$ for all p in M.

An interesting question in differential geometry, which also occurs in the theory of partial differential equations is the following one: Is it possible to find for all p in M a submanifold $N(p)$ of M such that for all $q \in N(p)$ $T_q N(p) = \Delta(q)$? Before we can give the general solution we first give some definitions.

DEFINITION 2.3. Let $\Delta$ be a k-dimensional $C^\infty$-distribution on M. A (k-dimensional) submanifold N of M is called an integral manifold of $\Delta$ if for all $p \in N$ we have $T_p N = \Delta(p)$, $\Delta$ has the integral manifold property if for all $p \in M$ there exist an integral manifold $N_p$.

DEFINITION 2.4. A k-dimensional $C^\infty$-distribution $\Delta$ on M is called involutive or integrable (see [2]) if for all $X,Y \in \Delta$ also $[X,Y] \in \Delta$.
Now we are able to give the solution of the above question:

THEOREM 2.5. *A k-dimensional* $C^\infty$ *distribution* $\Delta$ *on* M *has the integral manifold-property if and only if* $\Delta$ *is involutive.*

REMARK. A 1-dimensional distribution $\Delta$ is trivially involutive. Integral-manifolds can be found as integral curves of a non-zero vectorfield in $\Delta$. Theorem 2.5 is known as Froebenius' theorem, which also has a local version:

THEOREM 2.5'. *Let* $\Delta$ *be a k-dimensional* $C^\infty$ *distribution on* M. *If* $\Delta$ *is involutive then for every* $p \in M$ *there exists a coordinate system* $(x,U(p))$ *with*

$$x : U(p) \to \mathbb{R}^n \quad x(p) = 0, \quad x(U(p)) = (-\varepsilon,\varepsilon),\dots,(-\varepsilon,\varepsilon)\ (n\text{-}times)$$

*such that for each* $a_{k+1},\dots,a_n$ *with* $|a_j| < \varepsilon$, $j = k+1,\dots,n$ *the set* $\{q \in U(p) \,|\, x_{k+1}(q) = a_{k+1}(q),\dots,x_n(q) = a_n\}$ *is an integral manifold of* $\Delta$. *Moreover we can find vectorfields* $X_1,\dots X_k \in V^\infty(U(p))$ *such that in local coordinates* $x_i(q) = \frac{\partial}{\partial x_i}\Big|_{x(q)}$ $\quad i = 1,\dots,k$.

This means that locally we can find a set of vectorfields $X_1,\dots,X_k \in \Delta$ such that the corresponding integral curves will transform in the local chart into straight lines. The family of integral manifolds of M of theorem 2.5 is called a foliation of M. For a detailed study the reader is referred to the excellent portugesian book of LINS NETO & CAMACHO [4].
We now proceed with a few examples which will be important in the next section.

EXAMPLE 2.6. $M = \mathbb{R}^n$. We use 'global' coördinates $(x_1\dots x_n)$. Consider the $C^\infty$ distribution $\Delta$ which in each point is spanned by $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\}$.
$\Delta$ is involutive. Every integral manifold N of $\Delta$ has the form

$$\{(x_1,\dots,x_n) \,|\, x_{k+1} = a_{k+1},\dots x_n = a_n\}.$$

In fact we made a decomposition if $\mathbb{R}^n$ in two subspaces: $\mathbb{R}^k \times \mathbb{R}^{n-k}$.
This turns out to be useful in the theory of (A,B)-invariant subspaces.
The following example will show that the situation is not always 'fine'.

EXAMPLE 2.7. [4] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) = \alpha(x^2)e^y$ where $\alpha : \mathbb{R} \to \mathbb{R}$ a $C^\infty$ function such that

$$\begin{aligned} &\alpha(t) = 1 \quad \text{for } t \in (-\varepsilon,\varepsilon) \\ &\alpha(1) = 0 \\ &\alpha'(t) < 0 \quad \text{for } |t| > \varepsilon. \end{aligned}$$

The integral manifolds of the 1-dimensional distribution $\Delta$ on $\mathbb{R}^2$ now are given by the level curves of f. We can define an equivalence relation $\sim$ on $\mathbb{R}^2$ by $p_1 \sim p_2 \Leftrightarrow p_1$ and $p_2$ are on the same integral manifold of $\Delta$. If we consider the quotient manifold $\mathbb{R}^2/_\sim$ we see that this space is not even a Hausdorff space. For example the points $\tilde{a} = \pi(1,t)$ and $\tilde{b} = \pi(1,t)$ don't have disjunct neighbourhoods (here $\pi : \mathbb{R}^2 \to \mathbb{R}^2/_\sim$ the quotient map).
The last example will illustrate that for global control theory we can sometimes only use local descriptions. SUSSMANN studied in [18] the problem whether or not we have a global decomposition as in example 2.6. Here we will not give the result, but we will proceed with the analytic analogue of Froebenius' theorem. Let $C^\omega(M)$ and $V^\omega(M)$ be defined as in the smooth case.

DEFINITION 2.8. A $C^\omega$-distribution $\Delta$ on M (from now on $C^\omega$) is a map $\Delta$ which assigns to each point $m \in M$ a linear subspace of $T_mM$ and such that for all m there exists a neighbourhood U(m) and $X_1,\dots,X_k \in V^\omega(M)$ such that $\Delta(q) = \text{Span}\{X_1(q),\dots,X_k(q)\}$ for all q in U(m). It is not necessarily true that $X_1,\dots,X_k$ are linear independent everywhere. The definitions of integral manifold and involutive remain the same as in the smooth case.

THEOREM 2.9. (NAGANO [5]) *An analytic distribution $\Delta$ on a $C^\infty$ manifold M has the integral manifold properly if and only if $\Delta$ is involutive.*

EXAMPLE 2.10. Let $\Delta$ be the distribution on $\mathbb{R}^2$ spanned by the vectorfield $(x,y) \to -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ ; the integral manifolds are the circles $\{x^2 + y^2 = r^2\}_{r\geq 0}$; so all integral manifolds are 1-dimensional except for the point (0,0).

## 3. CONTROL SYSTEMS ON MANIFOLDS

We start with some motivating examples.

EXAMPLE 3.1. Consider the linear system $\Sigma$: $\dot{x} = Ax+Bu$ with $x \in \mathbb{R}^n =: X$, $u \in \mathbb{R}^m =: U$ and A,B matrices of appropriate dimensions. Another way to look at $\Sigma$ is the following one, introduced by WILLEMS [15]. $\Sigma_X = \{\underline{x} : \mathbb{R} \to X \mid \underline{x}$ absolute continuous and there exists $\underline{u} : \mathbb{R} \to U$ such that $\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$ almost everywhere$\}$. This definition fits more to the general situation, but there are still problems in interpreting such expression on a manifold. The key-word seems to be trajectories. Of course we want a coordinate-free definition for a control system; there are no a priori coordinates on a manifold. Another point is that in the above definition the input space plays a rôle. The only requirement in $\Sigma_X$ is that we are interested in trajectories $\underline{x} : \mathbb{R} \to X$ with $\dot{\underline{x}}(t) - A\underline{x}(t) \in \operatorname{Im} B := \mathcal{B}$, $\forall t \in \mathbb{R}$. Of course $\mathcal{B}$ has to be identified with the corresponding linear subspace of $T_{\underline{x}(t)}X$. Finally an important observation is the fact that the definition of $\Sigma_X$ does not depend on the feedback-group [19] consisting of the following coordinate transformations:

1) $S \in Gl(n)$ $\quad (A,B) \mapsto (SAS^{-1}, SB)$
2) $Q \in Gl(m)$ $\quad (A,B) \mapsto (A,BQ)$
3) $F \in L(\mathbb{R}^m, \mathbb{R}^n)$ $\quad (A,B) \mapsto (A+BF,B)$.

The set of all trajectories of $\Sigma_X$ is feedback invariant.

EXAMPLE 3.2. ([20]) Consider a spherical pendulum with a gasjet control which is always directed in the tangent space. We suppose that the magnitude and direction of the jet is completely adjustable within the tangent space. It is easy to give a local description of this situation: $\dot{x}(t) = A(x(t)) + \sum_{i=1}^{2} u_i(t)\, X_i(x(t))$, where $A(\ )$, $X_1(\ )$ and $X_2(\ )$ are $C^\infty$. vectorfields on $S^2$ and $X_1(x) \neq X_2(x)$, $u = (u_1,u_2) \in \mathbb{R}^2$. But it is clear that this will not a global description, for every $C^\infty$ vectorfield $X_1$ on $S^2$ has a singular point p, i.e. $X_1(p) = 0$([2]). In p the controls form a 1-dimensional subspace of $T_pS^2$ which contradicts our assumption of free direction of the gas-jet.

The last example is of great importance; it shows that for defining general control systems (i.e. control systems on manifolds) we need another description then the often used $\dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} u_i(t)X_i(x(t))$, see WILLEMS [15] and BROCKETT [20].

Now we will give the definition of a control system on a manifold. Again, as in section 2, we have to distinguish between the smooth and the analytic case.

DEFINITION 3.3. An affine distribution $\Delta$ on M ($C^\infty$ or $C^\omega$ manifold) will be a map $\Delta$ which assigns to each point m in M an affine subspace of $T_mM$. $\Delta$ is k-dimensional if the affine subspace $\Delta(m)$ is k-dimensional for all m.

DEFINITION 3.4. A $C^\infty$ k-dimensional control system on a $C^\infty$ manifold M will be a k-dimensional affine distribution $\Delta$ on M such that for all m there is a neighbourhood U(m) and vectorfields $X_0,\ldots,X_k \in V^\infty(M)$ such that

$$\forall q \in U(m) \ : \ \Delta\ (q) = X_0(q) + \mathrm{Span}\{X_1(q),\ldots,X_k(q)\}.$$

And in the analytic case:

DEFINITION 3.5. A $C^\omega$ control system on M ($C^\omega$) will be an affine distribution $\Delta$ on M such that for all m there is a neighbourhood U(m) and vectorfields $X_0,\ldots,X_k \in V^\omega(M)$ with the property that $\forall q \in U(m)$:

$$\Delta(q) = X_0(q_1) + \mathrm{Span}\{X_1(q),\ldots,X_k(q)\}.$$

We want to make some remarks about these definitions:

i) Although it is not necessary that in the $C^\infty$-case we have fixed dimension, we made this assumption for simplicity. It turns out to be extremely difficult to get results without this condition. (Of course we claim that the $X_i$'s in definition 3.4 are independent on U(m), $i = 1,\ldots,k$). Also we can give a $C^r$ ($r \geq 0$) version of a control system but again it makes it more difficult to treat.

ii) Locally we arrived at the 'famous' nonlinear situation $\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{k} u_i(t)\ X_i(x(t))$ as studied in for example [13], [11], but it is important to point out that we have a 'feedback invariant' form, for if we choose functions $\alpha_i$ ($i = 1,\ldots,k$) $\in C(M)$ then

$$\Delta(q) = X_0(q) + \mathrm{Span}\{X_1(q),\dots,X_k(q)\}$$

$$= X_0(q) + \textstyle\sum_{i=1}^{k} \alpha_i(q)\ X_i(q) + \mathrm{Span}\{X_1(q),\dots,X_k(q)\}$$

iii) As already noted in the examples in the beginning of this section we only have to do with the effects caused by the inputs (in linear terms we do not see U, the input-space, but only Im B). One can even say that the inputs are parametrized by the choice of the vectorfields $X_1,\dots,X_k$.

EXAMPLE 3.6. $M = \mathbb{R}^n$.

We use global coordinates $(x_1,\dots,x_n)$ and consider the linear system $\sum : \dot{x} = Ax + Bu \quad u \in \mathbb{R}^m$ (A and B of appropriate dimensions). $\sum$ can also be defined by an affine distribution $\Delta\ x \to Ax + \mathrm{Span}\{b_1,\dots,b_m\}$ where $b_1,\dots,b_m$ are the columns of Im B.

In this example we still used the standard coordinates for $\mathbb{R}^n$. As already said we donot want to use a coordinate representation in describing a system. The definitions 3.4 and 3.5 are also coordinateless. But this raises the question whether or not a system on $\mathbb{R}^n$ is linear. Therefore we use a recent result of JAKUBCZYK & RESPONDEK [9] (see also [10]). Although they give the result on $\mathbb{R}^n$ it is easy to formulate it in general terms. Let $\Delta$ be a $C^\infty$ control system on M. By $\Delta_0 =: \Delta - \Delta$ we denote all $B \in V^\infty(M)$ with $B = X - Y$, $X,Y \in \Delta$ (Recall that $X \in \Delta$ means $X \in V^\infty(M)$ and $X(p) \in \Delta(p)\ \forall p \in M$). We also observe that $\Delta + \Delta_0 = \Delta$. In the linear case $\Delta_0$ stands for Im B. We define $\Delta_k =: [\Delta,\Delta_{k-1}]$, which means for example $Y \in \Delta_1$. then Y is the Liebracket of a vectorfield $X \in \Delta$ and a vectorfield $B \in \Delta_0$. Again we wish to point out the linear analogue where $\Delta_1 = \mathcal{B} + A\mathcal{B}$, $\Delta_2 = \mathcal{B} + A\mathcal{B} + A^2\mathcal{B}$ ... . Whether or not a system is linear now can be expressed in terms of $\Delta$ and $\Delta_i$ $(i \in \mathbb{N})$.

DEFINITION 3.7. We will call a control system on a manifold M locally linearizable if for each point m we can give a coordinate neighbourhood $(x,\mathcal{U})$ such that in these coordinates the system has the form $\dot{x} = A_x + Bu + \xi$ $\xi \in \mathbb{R}^n$, fixed. Now we are able to apply the result of Jakubczyk and Respondek (we only give a reformulation).

THEOREM 3.8. ([9]) *Let* $\Delta$ *be a* $C^\infty$ *control system on* M. $\Delta$ *is locally linearizable if and only if*

1) $[\Delta_k,\Delta_\ell] \subseteq \max\{\Delta_k,\Delta_\ell\} \quad \forall k,\ell \in \mathbb{N}$,
2) $\forall k \in \mathbb{N}$ $\dim(\Delta_i(x))$ *is independent of* x,
3) $\dim \Delta_{n-1} = n$ (= dim M).

REMARKS.

i) The theorem only has to do with the controllable situation, see 3.

ii) There exists a direct connection between the dimensions of $\Delta_i$ and the Kronecker indices of the pencil (A,B), the resulting linear system is in BRUNOVSKY - canonical form ([19]): define

$$\dim \Delta_i = r_i \qquad i \in \mathbb{N}$$

$$p_0 =: r_o,\ p_i =: r_i - r_{i-1}$$

then the $p_i$'s are the Kronecker indices of the pair (A,B).

iii) The conditions 1 and 2 of theorem 3.8 have an interesting consequence in terms of distributions. It is easy to see that the distribution $\Delta_i$ is involutive, so for all $i \in \mathbb{N}$ we can apply Froebenius' theorem (theorem 2.5). Furthermore the distributions are nested: $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \ldots$, which is the essential part of the proof.

iv) With great ease we can apply the results of [14] to this situation. Structural stability of a system depends on the $\Delta_i$'s.

v) It is easy to construct a $C^\omega$-example which does not satisfy the conditions of theorem 3.8. In fact every bilinear system does not satisfy the dimension assumption.

Let

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u_1 \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then we see that $\Delta_0$ is given by

$$\Delta_0(x_1,x_2) = \mathrm{Span}\{x_2 \frac{\partial}{\partial x_1}\Big|_{\underline{x}}, \frac{\partial}{\partial x_2}\Big|_{\underline{x}}\},$$

$\Delta_0$ is not involutive.

It is an interesting question whether or not a $C^{\omega}$ analogue of theorem 3.8 exists.

vi) Another observation of this theorem is that the feedback-group now changes; instead of linear diffeomorphisms on $\mathbb{R}^n$, i.e. elements of $Gl(n)$, we can use all diffeomorphisms on the state space $\mathbb{R}^n$.

Next we will give an elementary example of the above theorem, which illustrates how to linearize.

EXAMPLE 3.9. Consider the nonlinear system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ 1 \end{pmatrix} u \quad \text{on } \mathbb{R}^2 .$$

$\Delta_0$ is spanned by the vectorfield $B(x_1,x_2) = 2x_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ and $\Delta_0$ has fixed dimension. Solving the associate differential equation to get the integral curves of B (which are also the integral manifolds of the distribution $\Delta_0$) leads to $x_1 - x_2^2$ = constant. We now compute a specific new member of $\Delta_1$:

$$[\Delta,\Delta \quad \left[ x_2 \frac{\partial}{\partial x_2} , 2x_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right] = - \frac{\partial}{\partial x_1} ,$$

and furthermore $\Delta_1$ has fixed dimension. Solving again the associate differential equation now gives $x_2$ = constant. Finally we use the coordinate transformation

$$\begin{cases} \tilde{x}_1 = x_1 - x_2^2 \\ \tilde{x}_2 = x_2 \end{cases}$$

and we arrive at the linear control system

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Of course we could have used the transformation

$$\begin{cases}\tilde{x}_1 = x_1 - x_2^2 + \beta \\ \tilde{x}_2 = x_2 + \alpha\end{cases}$$

then we'll get the system

$$\begin{pmatrix}\tilde{x}_1 \\ \tilde{x}_2\end{pmatrix}' = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{x}_1 \\ \tilde{x}_2\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix} u - \begin{pmatrix}0 \\ \alpha\end{pmatrix}$$

or, there is no reason of selecting a specific point of $\mathbb{R}^2$ as the origin of the new coordinate system, what also can be expressed by saying that the feedback group is extended by translations.

Controllability of a control system.
The last decade various people attacked the problem of controllability of nonlinear systems (see e.g. [6], [7], [8]). Before we can give the result in our notation we have to introduce the concept of accessibility.

<u>DEFINITION 3.10.</u> ([7]) Let $\Delta$ be a control system on M. Given a subset O of M a point x' is weak O-accessible from a point x" (denoted by $x'WA_Ox''$) if there exists a collection of vectorfields $X_1,\dots,X_\alpha \in \Delta$ and points $x' = x_0,x_1,\dots,x_\alpha = x''$ such that $x_i$ belongs to the integral curve of $X_i$ through $x_{i-1}$ $(i = 1,\dots,\alpha)$ and the paths, given by these integral curves, belong to O. For a neighbourhood O of $x_0$ the set of all weak-O-accessible points from $x_0$ is denoted by $WA_O(x_0)$.

<u>DEFINITION 3.11.</u> ([7]) $\Delta$ is locally weakly controllable in $x_0$ if $WA_O(x_0)$ is a neighbourhood of $x_0$ for all O. $\Delta$ is locally weakly controllable if it is locally weakly controllable in $x_0$ for all $x_0$ in M. Recall that $\Delta_0 = \Delta - \Delta = \{X-Y \mid X,Y \in \Delta\}$ and $\Delta_k = [\Delta,\Delta_{k-1}]$.

<u>THEOREM 3.12.</u> ($C^\infty$ version) *Let* $\Delta$ *be a* $C^\infty$ *control system on* M. *Let* $\Delta_p = \Delta_{p-1}$ *be a* k*-dimensional involutive distribution on* M. *Then for all* $x_0$ *in* M $WA_O(x_0)$ *is an open subset of the corresponding integral manifold through* $x_0$.

<u>THEOREM 3.12.</u> ($C^\omega$ version) *Let* $\Delta$ *be a* $C^\omega$ *control system on* M. *Let* $\Delta_p = \Delta_{p-1}$ *be an involutive distribution on* M. *Then for all* $x_0$ *in* M $WA_O(x_0)$ *is an open*

*subset of the corresponding integral manifold through* $x_0$.

The proof of this theorem may be found in the literature ([7]).

COROLLARY. $\Delta$ *is locally weakly controllable if* $\dim \Delta_\infty = \dim \Delta_k = n$ (k *sufficiently large*).

REMARK. We note that although it seems to be an infinite procedure to compute $\Delta_\infty$ we can stop after a finite number of times ($\Delta$ is locally of the form $X_0 + \mathrm{Span}\{X_1,\dots,X_\alpha\}$). In the controllable case for example we have done after (n-1) steps (compare with the linear case).

## 4. $(\Delta,\Delta_0)$ INVARIANT DISTRIBUTIONS

In this section we want to discuss the generalized notion of (A,B)-invariance. Recently several people studied this problem (ISODORI et al [11], NOMURA & FORUTA [12], HIRSCHORN [13]). Although we don't consider output in this paper (so we cannot apply the results to the disturbance decoupling problem) the nonlinear analogue of (A,B)-invariance is interesting to treat with the distributional approach.

In [17] WILLEMS has given a collection of various definitions of (A,B)-invariance in the linear case. We'll pick up a few of them which turn out to be most useful for nonlinear systems. Let

$$\textstyle\sum : \dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n =: \mathcal{X}, \quad u \in \mathbb{R}^m =: \mathcal{U}.$$

DEFINITION 4.1. (1) A linear subspace $\mathcal{V} \subset \mathcal{X}$ is (A,B)-invariant if there exists a (linear) feedback $F : \mathcal{X} \to \mathcal{U}$ such that $A_F\, \mathcal{V} \subset \mathcal{V}$ where $A_F =: A+BF$.

DEFINITION 4.1. (2) A linear subspace $\mathcal{V} \subset \mathcal{X}$ is (A,B)-invariant if $A\mathcal{V} \subset \mathcal{V}+B$.

DEFINITION 4.1. (3) A linear subspace $\mathcal{V} \subset \mathcal{X}$ is (A,B)-invariant if $\sum(\mathrm{mod}\ \mathcal{V})$ is a linear system. We now give the distributional version of this definition (the reader is referred to example 2.6). Let $\mathcal{V}$ be a linear subspace of $\mathcal{X}$. We can associate a distribution $D_{\mathcal{V}}$ with the linear subspace $\mathcal{V}$ by defining $D_{\mathcal{V}}(x) = \mathcal{V} \subseteq T_x\mathbb{R}^n$ where we use the natural identification of $\mathbb{R}^n$ with

$T_x\mathbb{R}^n$. Another way of defining $D_V$ is given by the following: Let $\{v_1,\ldots,v_k\}$ be an orthonormal basis of $V$ then $D_V$ is given by $\mathrm{Span}\{\frac{\partial}{\partial v_1},\ldots,\frac{\partial}{\partial v_k}\}$.
The condition $A_F V \subset V$ will transform in

$$[A_F, \frac{\partial}{\partial v_i}](\underline{x}) \in \mathrm{Span}\{\frac{\partial}{\partial v_1}\Big|_{\underline{x}},\ldots,\frac{\partial}{\partial v_k}\Big|_{\underline{x}}\} \quad \forall i=1,\ldots,k$$

$\Longleftrightarrow$ $A_F$ has the form (with respect to a basis $\{v_1,\ldots,v_k,\ldots v_n\}$)

$$A_F = \begin{array}{c} \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline \underbrace{0}_{k} & A_{22} \end{array}\right) \end{array} \begin{array}{l} \}\ k \\ \\ \end{array} \qquad (*)$$

Now we will give generalization of 4.1. We don't distinguish between the smooth and the analytic case. In the context of the definition of a control system (A,B)-invariance becomes $(\Delta,\Delta_0)$-invariance.

<u>DEFINITION 4.2</u>. An involutive distribution D (fixed dimension) on M will be called $(\Delta,\Delta_0)$-invariant if there exists X in $\Delta$ such that $[X,D] \subseteq D$. If we work out a coordinate version of this definition then we get the following appealing result ($C^\infty$ version). Locally we can find around each point in M a coordinate system such that the involutive distribution D - with fixed dimension p - is spanned by the vectorfields $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_p}$ (theorem 2.5'). Writing down the equation $[X,D] \subseteq D$ now gives that

$$\frac{\partial X_i(x)}{\partial x_j} = 0 \quad \begin{array}{l} \forall i = p+1,\ldots,n \\ \forall_j = 1,\ldots,p \end{array} \quad \text{if } X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i},$$

or equivalently if we write $\underline{x}_1 = (x_1,\ldots,x_p)$ and $\underline{x}_2 = (x_{p+1},\ldots,x_n)$ then we get the following form for X

$$X(\underline{x}) = \begin{pmatrix} X_1\ (\underline{x}_1,\underline{x}_2) \\ \vdots \\ X_p\ (\underline{x}_1,\underline{x}_2) \\ X_{p+1}(\underline{x}_2) \\ \vdots \\ X_n\ (\underline{x}_2) \end{pmatrix} \quad \text{which is the nonlinear analogue of } (*)!$$

Next we will show that under certain conditions the equivalence of definition 4.1 (1) and definition 4.1 (2) will be true in the nonlinear case. We will treat here the $C^\infty$ version although we also can do in the $C^\omega$-case.

ASSUMPTION. *From now on we consider a class of involutive distributions,* $F(\Delta_0)$ *(here we use* $F(\Delta_0)$ *which stands for a 'friend' of* $\Delta_0$*, compare* [16]) *such that* $D \in F(\Delta_0) \Longleftrightarrow \Delta_0 + D$ *is involutive and has fixed dimension.*

REMARK. For a linear system $\overset{\circ}{x} = Ax + Bu$ one only considers $(A,B)$-invariant subspaces $V$ which are linear subspaces of $\mathbb{R}^n$; moreover Im $B$ is a linear subspace of $\mathbb{R}^n$. The associated distribution automatically satisfies the above property (and also $\Delta_0$ is involutive).

THEOREM 4.3. $D \in F(\Delta_0)$ *is locally* $(\Delta,\Delta_0)$*-invariant if and only if* $[\Delta,D] \subseteq D + \Delta_0$.

PROOF. ($\Rightarrow$) $D$ is $(\Delta,\Delta_0)$ invariant implies that there exists $X \in \Delta$ such that $[X,D] \subseteq D$. Then, for every $\tilde{X} \in \Delta$ we have

$$[\tilde{X},D] = [X,D] + [\tilde{X}-X,D] \subseteq D + [\Delta_0,D] \subseteq D + \Delta_0.$$

(The last inclusion follows from the fact that $D \in F(\Delta_0)$).
($\Leftarrow$) Now we assume $[\Delta,D] \subseteq D + \Delta_0$. Let $X \in \Delta$ then $[X,D] \subseteq D + \Delta_0$.
We now construct the 'feedback' (associated with the choice of $X$).
Let $D$ be a $k$-dimensional involutive distribution. So around each point $p \in M$ we can find a local chart $(U(p),x)$ and vectorfields $Y_1,\ldots,Y_k$ on $U(p)$ as in the local Froebenius' theorem (Th 2.5') $[X,D] \subseteq D + \Delta_0 \Rightarrow \exists$ vectorfields $B_1,\ldots,B_k \in \Delta_0$ such that $[Y_i,X] = B_i$ (mod $D$) $i = 1,\ldots,k$ (here mod $D$ means of course modulo a vectorfield in $D$).

$$\Rightarrow \quad [Y_j,[Y_i,X]] = [Y_j,B_i] \ (\text{mod } D) \qquad i,j = 1,\ldots,k$$

$$[Y_i,[Y_j,X]] = [Y_i,B_j] \ (\text{mod } D) \qquad i,j = 1,\ldots,k$$

$$[Y_j,[Y_i,X]] - [Y_i,[Y_j,X]] = [Y_j,B_i] - [Y_i,B_j] \ (\text{mod } D) \qquad i,j = 1,\ldots,k$$

but by the Jacobi-identity we have

$$[Y_j,[Y_i,X]] - [Y_i,[Y_j,X]] = [[Y_j,Y_i],X],$$

and by the choice of $Y_1,\dots,Y_k$ we have $[Y_i,Y_i] = 0$

$$\Rightarrow \qquad [Y_j,B_i] - [Y_i,B_j] = 0 \pmod{D} \qquad i,j = 1,\dots,k. \qquad (*)$$

Now we construct a vectorfield B in $\Delta_0 + D$ such that $[Y_i,B] = B_i \pmod{D}$.
In the local chart $(U(p),x)$ we let

$$B_i(x) = \sum_{j=1}^{n} B_i^j(x) \frac{\partial}{\partial x_j}\Big|_{\underline{x}}$$

and we define

$$\beta_i : \mathbb{R}^n \to \mathbb{R}^n \text{ by } \beta_i(x) = \begin{pmatrix} B_i^1(x) \\ \vdots \\ B_i^n(x) \end{pmatrix} \qquad i = 1,\dots,k.$$

Define $\beta : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\beta(0,\dots 0,x_{k+1},\dots x_n) = 0$$

$$\begin{aligned}\beta(x_1,\dots,x_n) = &\int_0^{x_1} \beta_1(t,0\dots,0,x_{k+1},\dots,x_n)dt + \\ &\int_0^{x_2} \beta_2(x_1,t,0,\dots 0,x_{k+1},\dots,x_n)dt + \\ &\dots \\ &+ \int_0^{x_k} \beta_k(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_n)dt\end{aligned}$$

and B will be the corresponding vectorfield. In the local coordinates we compute $[Y_i,B]$: i.e.

$$\frac{\partial B}{\partial x_i}(x_1,\dots,x_n) = \beta_i(x_1,\dots x_i,0\dots 0,x_{k+1},\dots,x_n)$$

$$+ \frac{\partial}{\partial x_i}\int_0^{x_{i+1}} \beta_{i+1}(x_1,\dots,x_i,t,0\dots 0,x_{k+1},\dots,x_n)dt$$

$$\dots$$

$$+ \frac{\partial}{\partial x_i}\int_0^{x_k} \beta_k(x_1\dots x_{k-1},t,x_{k+1}m\dots,x_n)dt.$$

Now from (*) we have

$$\frac{\partial\beta_i}{\partial x_j}(x) - \frac{\partial\beta_j}{\partial x_i}(x) = 0 \pmod D.$$

So we have

$$\frac{\partial\beta}{\partial x_i}(x_1,\dots,x_n) = \beta_i(x_1\dots x_i,0\dots 0,x_{k+1}\dots x_n)$$

$$+ \int_0^{x_{i+1}} \frac{\partial\beta_i}{\partial t}(x_1\dots x_i,t,0\dots,x_{k+1}\dots x_n)dt$$

$$\dots$$

$$+ \int_0^{x_k} \frac{\partial\beta_i}{\partial t}(x_1\dots x_{k-1},t,x_{k+1}\dots x_n)dt$$

$$+ D_1(x_1\dots x_n).$$

Here $D_1(x_1\dots x_n)$ is a vectorfield which by the involutivity of D belongs to D. You'll get it by integrating the differences $\frac{\partial\beta_i}{\partial x_j} - \frac{\partial\beta_j}{\partial x_i}$ in D by a vectorfield in D ($\int_0^{x_j}\dots dx_j$ is just integrating with respect to the vectorfield $Y_j$)

$$\Rightarrow \quad \frac{\partial\beta}{\partial x_i}(x_1\dots x_n) = \beta_i(x_1\dots x_i,0\dots 0,x_{k+1}\dots x_n)$$

$$+ \beta_i(x_1\dots x_{i+1},0\dots 0,x_{k+1}\dots x_n) - \beta_i(x_1\dots x_1,0\dots 0,x_{k+1}\dots x_n)$$

$$\dots$$

$$+\beta_i(x_1\ldots x_n) - \beta_i(x_1\ldots x_{k-1},0,x_{k+1}\ldots x_n) \pmod D$$

$$= \beta_i(x_1\ldots x_n) \pmod D.$$

Finally we observe that the vectorfield B belongs to $\Delta_0 + D$. (By the involutivity of $\Delta_0 + D$ and because we integrate with respect to vectorfields from D!) Now we can see that B is the appropriate feedback for

$$[Y_i,X] = B_i \pmod D = [Y_i,B] \pmod D$$

$\Rightarrow [Y_i,X-B] \in D$ and now $X-B \in \Delta$ if $B \in \Delta_0$ (otherwise we only use the component of B in $\Delta_0$) $\Rightarrow [D,X-B] \subset D$. □

REMARKS.

i) We have proved the $C^\infty$-version here, in the same spirit we can prove the $C^\omega$-case.

ii) Feedback is unique up to vectorfields which belong to $D \cap \Delta_0$ (Compare with the linear case).

iii) We can also drop the assumptions about $\Delta_0$ and D. In the same way we can prove the following theorem: D is $(\Delta,\Delta_0)$-invariant $\iff [\Delta,D] \subseteq [\Delta_0,D]+D$, but the 'feedback' we can construct, belongs to involutive closure of $\Delta_0 + D$ (which is not a feedback in the usual sense).

iv) It is straightforward to show that - under the assumption that $\Delta_0$ is involutive - our results are a generalization of [13]. The distributional approach presented here, seems to be better in treating nonlinear (A,B)-invariance.

COROLLARY 4.4.

i) *If* $D_1,D_2 \in \mathcal{F}(\Delta_0)$ *then* $\overline{D_1+D_2}$ *- the involutive closure of the distribution* $D_1+D_2$ *- belongs to* $\mathcal{F}(\Delta_0)$. *So* $\mathcal{F}(\Delta_0)$ *is closed under addition.*

ii) *If* $D_1,D_2$ *are* $(\Delta,\Delta_0)$*-invariant distributions then* $\overline{D_1+D_2}$ *is* $(\Delta,\Delta_0)$*-invariant*

PROOF.

i) We have to show that $\overline{D_1+D_2+\Delta_0}$ is involutive. It will be clear that we've done if $\forall Y \in \overline{D_1+D_2}$, $\forall B \in \Delta_0$, $[Y,B] \in \overline{D_1+D_2+\Delta_0}$.

$Y \in \overline{D_1+D_2}$ then $Y \in D_1$ (or $Y \in D_2$) or $Y \in [D_1,D_2]$ or inspaces generated by higher order Liebrackets.

$Y \in D_1 \Rightarrow [Y,B] \in D_1+\Delta_0 \subset D_1+D_2+\Delta_0$

$Y = [Y_1,Y_2]$, $Y_1 \in D_1$, $Y_2 \in D_2$, then

$[Y,B] = [[Y_1,Y_2],B] = [Y_1,[Y_2,B]] + [Y_2,[B,Y_1]]$ (Jacobi!) and this belongs to $\overline{D_1+D_2}+\Delta_0$. In the same way one can treat higher order brackets.

ii) $[\Delta,D_1] \subseteq D_1+\Delta_0$

$[\Delta,D_2] \subseteq D_2+\Delta_0$.

Thus $[\Delta,\overline{D_1+D_2}] \subseteq \overline{D_1+D_2}+\Delta_0$

$\Rightarrow \overline{D_1+D_2}$ is $(\Delta,\Delta_0)$-invariant.

Finally we note that $\overline{D_1+D_2}$ as well as $\overline{D_1+D_2}+\Delta_0$ have fixed dimension if $D_1,D_2 \in F(\Delta_0)$.

LEMMA 4.5. *Let $F$ be a non-empty class of involutive distributions on $M$ such that also $D \in F \Rightarrow D$ has fixed dimension. Then $F$ contains a supremal element $D^*$ (i.e. $\forall D \in F : D \subseteq D^*$).*

PROOF. By the fact that $F$ is closed under addition there exists an involutive distribution of greatest dimension: $D^* \in F$. Now, if $D \in F$ we have $\overline{D+D^*} \in F$ and so $\dim(D^*) \geq \dim(\overline{D+D^*})$ that is, $D^* = \overline{D+D^*}$, hence $D^* \supset D$ and so $D^*$ is supremal. □

THEOREM 4.6. *Every $K \in F(\Delta_0)$ contains a unique supremal $(\Delta,\Delta_0)$-invariant distribution. We will denote this distribution by $S(\Delta,\Delta_0;K)$.*

PROOF: Corollary 4.4 and lemma 4.5. □

We want to conclude this paper with an algorithm for $S(\Delta,\Delta_0;K)$. One should compare this procedure with the linear algorithm (See [16]). Define

$$\Delta^{-1}(\Delta_0+D) = \{X \in V(M) \mid [\Delta,X] \subseteq \Delta_0+D\}.$$

THEOREM 4.7. *Let $K \in F(\Delta_0)$. Define the sequence $\{D^\mu\}_{\mu=0,1,2,\ldots}$ according to*

$$D^0 = K$$
$$D^\mu = K \cap \Delta^{-1}(\Delta_0 + D^{\mu-1}) \qquad \mu = 1,2,\ldots$$

Then i) $D^\mu \subset D^{\mu-1}$ $\quad \mu = 1,2,\ldots$

ii) $D^\mu$ is involutive and moreover if we assume that $D^\mu$ has fixed dimension then $D^\mu \in \mathcal{F}(\Delta_0)$ $(\mu = 0,1,2,\ldots)$.

iii) for some $k \leq \dim(K)$ we have $D^k = \sup(\Delta,\Delta_0;K)$.

PROOF.

i) $D^\mu \subset D^{\mu-1}$

Clearly $D^1 \subset D^0$ and if $D^\mu \subset D^{\mu-1}$ then

$$D^{\mu+1} = K \cap \Delta^{-1}(\Delta_0+D^\mu) \subseteq K \cap \Delta^{-1}(\Delta_0+D^{\mu-1}) = D^\mu$$

ii) $D^\mu$ is involutive.

$D^0 = K \in \mathcal{F}(\Delta_0)$ thus $D^0$ is involutive.

Suppose $D^{\mu-1} \in \mathcal{F}(\Delta_0)$ then $D^\mu = K \cap \Delta^{-1}(\Delta_0+D^{\mu-1})$,

now

$$\left.\begin{array}{r} X,Y \in D^\mu \Rightarrow X,Y \in K \\ K \in \mathcal{F}(\Delta_0) \end{array}\right\} \Rightarrow [X,Y] \in K \tag{1}$$

and $\forall A \in \Delta$ we have $[A,X] \in \Delta_0+D^{\mu-1}$

$[A,Y] \in \Delta_0+D^{\mu-1}$

$$\begin{aligned} [A,[X,Y]] &= -[X,[Y,A]] - [Y,[A,X]] \quad \text{(Jacobi)} \\ &\in [D^\mu,\Delta_0+D^{\mu-1}] \\ &\in [D^{\mu-1},\Delta_0+D^{\mu-1}] \subseteq \Delta_0+D^{\mu-1} \end{aligned} \tag{2}$$

(here we use $\Delta_0+D^{\mu-1}$ is involutive).

From (1) and (2) we conclude that $D^\mu$ is involutive. By the dimension assumption and the next lemma (4.8) it will follow that $D^\mu+\Delta_0$ is involutive and $D^\mu \in \mathcal{F}(\Delta_0)$.

iii) Suppose $D \in \mathcal{F}(\Delta_0)$ $D \subset K$ and $D$ is $(\Delta,\Delta_0)$-invariant then $D \subset K$ and $[\Delta,D] \subseteq D+\Delta_0$

$\Longleftrightarrow D \subset K, D \subseteq \Delta^{-1}(D+\Delta_0)$

$\Rightarrow D \subset D^0$ and if $D \subset D^{\mu-1}$ then

$$D \subset K \cap \Delta^{-1}(D+\Delta_0) \subset K \cap \Delta^{-1}(D^{\mu-1}+\Delta_0) = D^\mu.$$

Therefore we have $D \subset D^k$ $\forall k \in \mathbb{N}$. But we can easy show by a dimension argument that $D^{\mu+1} = D^{\mu}$ for $\mu \geq \dim(K)$. Therefore $\lim_{\mu\to\infty} D^{\mu}$ exists and equals $S(\Delta,\Delta_0;K)$. □

We still have to show that $D^{\mu}+\Delta_0$ is involutive.

LEMMA 4.8. *Let $D_1,D_2$ and $D_3$ be involutive distributions of fixed dimension and $D_1 \subset D_2$. $D_2+D_3$ is involutive and has fixed dimension then $D_1+D_3$ is involutive and has fixed dimension.*

PROOF. We only give here the proof in case $D_2 \cap D_3 = \underline{0}$; the general case can be done in the same wat. By a modification of Frobenius' theorem, as given in [9], we can locally find vectorfields $X_1,\ldots,X_m$ such that $[X_i,X_j] = 0$ $i,j = 1,\ldots,m$ and

$$\mathrm{Span}\{X_1\ldots X_k\} = D_1$$
$$\mathrm{Span}\{X_1\ldots,X_k\ldots,X_\ell\} = D_2$$
$$\mathrm{Span}\{X_1\ldots,X_k\ldots,X_\ell,\ldots X_m\} = D_2+D_3$$
$$D_1+D_3 = \mathrm{Span}\{X_1\ldots,X_k,X_{\ell+1}\ldots X_m\} \qquad \text{(here we use } D_2 \cap D_3 = \underline{0}\text{ !)}$$

and also $[X_i,X_j] = 0$ $i,j = 1,\ldots k,\ \ell+1,\ldots m$

$\Rightarrow D_1+D_3$ is involutive. □

An induction argument now will give that $D^{\mu}+\Delta_0$ is involutive for all $\mu \in \mathbb{N}$ (note that $D^0+\Delta_0 = K+\Delta_0$ is involutive).

## 5. CONCLUSION

In this paper we have attempted to give a new treatment of a particular class of nonlinear control systems. Under certain conditions we completely solved the problem of controlled-invariance. Apparently one can also consider some other 'linear problems' as for example: controllability subspaces in this terminology.

## ACKNOWLEDGEMENT

I would like to thank Arjan van der Schaft, Jan Willems and Jan van Schuppen for many useful discussions.

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